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The optimal saving with mixed parameters

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Abstract

This paper proposes two mixed models to study optimal saving in the presence of two types of risk: income risk and background risk. In the first model the income risk is a fuzzy number and the background risk is a random variable. In the second model the income risk is a random variable and the background risk is a fuzzy number. For these models three notions of precautionary saving are defined as indicators of the changes induced by the income risk and the background risk on the choice of optimal saving.

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The saving under uncertainty is a topic introduced in the economic research by Leland [1], Sandmo [2] and Drèze and Modigliani [3]. In paper [4], Kimball connects this topic with the concept of "prudence". These papers investigate the way the presence of risk modifies the saving. The change of optimal saving by adding risk elements is measured by the notion of precautionary saving. As a rule, necessary and sufficient conditions are looked for to have positive precautionary saving. In interpretation, positive precautionary saving means that in the presence of risk the consumer increases his level of optimal saving.

On the other hand, several authors [5], [6], [7] studied economic decision processes governed by two types of risk: primary risk (income risk) and background risk (labor income risk, loss of employment, divorce, etc.). The presence of background risk influences the choice of the optimal solution in the economic decision (see e. g. [8]).

The way the two types of risk act on the optimal saving was studied by Courbage and Rey [9] and Menegatti [10]. The influence of the primary risk and background risk is measured in [10] by precautionary saving and two-source precautionary saving. The main results of [10] offer necessary and sufficient conditions for the positivity of these two indicators.

All the above approaches are probabilistic. Risk can be also modeled by Zadeh's possibility theory [11]. In this case, risk will be mathematically described by possibilistic distributions (particularly by fuzzy numbers). Possibilistic models of risk can be found in [12], [13].

This paper proposes two mixed models for the study of the behaviour of the optimal saving under the action of the two types of risk. In the first model, the income risk is a fuzzy number and the background risk is a random variable.

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In the second model the situation is inverse: the income risk is a random variable and the background risk is a fuzzy number.

Following the line of [10], for both models three notions of precautionary saving are defined (of type *I* and *II*). These indicators measure the way the mixed representations of the two types of risk produce changes on the optimal saving. The main theorems of the paper state necessary and sufficient conditions on the indicators (inside each model).

We describe shortly the content of the paper. Section 1 presents fuzzy numbers and some of their indicators (by [14], [15], [12]) and Section 2 recalls from [13], [16] the notion of mixed expected utility and two important properties of it. A brief description of probabilistic models of optimal saving from [9], [10] is in Section 3.

Section 4 studies a mixed model of optimal saving in which the income risk is a fuzzy number and the background risk is a fuzzy variable. Section 5 deals with the second model of optimal saving, in which the income risk is a random variable and the background risk is a fuzzy number. The paper ends with conclusions.

1. Preliminaries on fuzzy numbers

Let X be a non-empty set of states. A *fuzzy subset* of X is a function $A : X \rightarrow [0, 1]$. A fuzzy set A is *normal* if $A(x) = 1$ for some $x \in X$. The support of A is defined by $\text{supp}(A) = \{x \in X | A(x) > 0\}$.

Next assume that $X = \mathbf{R}$. For $\gamma \in [0, 1]$, the γ -level set $[A]^\gamma$ of A is defined by

$$[A]^\gamma = \begin{cases} \{x \in \mathbf{R} | A(x) \geq \gamma\} & \text{if } \gamma > 0 \\ \text{cl}(\text{supp}(A)) & \text{if } \gamma = 0 \end{cases}$$

($\text{cl}(\text{supp}(A))$ is the topological closure of $\text{supp}(A)$.)

The fuzzy set A is *fuzzy convex* if $[A]^\gamma$ is a convex subset of \mathbf{R} for all $\gamma \in [0, 1]$.

A fuzzy subset A of \mathbf{R} is called a *fuzzy number* if it is normal, fuzzy convex, continuous and with bounded support.

If A, B are fuzzy numbers and $\lambda \in \mathbf{R}$ then the fuzzy numbers $A + B$ and λA are defined by

$$(A + B)(x) = \sup_{y+z=x} \min(A(y), B(z))$$

$$(\lambda A)(x) = \sup_{\lambda y=x} A(y)$$

A non-negative and monotone increasing function $f : [0, 1] \rightarrow \mathbf{R}$ is a *weighting function* if $\int_0^1 f(\gamma) d\gamma = 1$.

We fix a weighting function f and a fuzzy number A such that $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$ for $\gamma \in [0, 1]$. Let $u : \mathbf{R} \rightarrow \mathbf{R}$ be a continuous function (interpreted as a utility function).

The possibilistic expected utility $E(f, u(A))$ is defined by:

$$E(f, u(A)) = \frac{1}{2} \int_0^1 [u(a_1(\gamma)) + u(a_2(\gamma))] f(\gamma) d\gamma \quad (1)$$

If u is the identity function then from (1) one obtains the f -weighted possibilistic expected value $E(f, u(A))$ [20]:

$$E(f, A) = \frac{1}{2} \int_0^1 [a_1(\gamma) + a_2(\gamma)] f(\gamma) d\gamma \quad (2)$$

If $u(x) = (x - E(f, A))^2$ then one obtains the f -weighted possibilistic variance [17]:

$$\text{Var}(f, A) = \frac{1}{2} \int_0^1 [(a_1(\gamma) - E(f, A))^2 + (a_2(\gamma) - E(f, A))^2] f(\gamma) d\gamma \quad (3)$$

When $f(\gamma) = 2\gamma$ for $\gamma \in [0, 1]$, $E(f, A)$ and $\text{Var}(f, A)$ are the possibilistic mean value and the possibilistic variance of [18].

2. Mixed expected utility

Mixed expected utility is a notion introduced in [16] to build a model of risk aversion with mixed parameters: some parameters are described by fuzzy numbers and others are described by random variables. This notion has been used in [19] to study mixed investment models with background risk.

Let X be a random variable w.r.t. a probability space (Ω, \mathcal{H}, P) . We will denote by $M(X)$ its expected value and by $\text{Var}(X)$ its variance. If $u : \mathbf{R} \rightarrow \mathbf{R}$ is a continuous function then $u(X) = u \circ X$ is a random variable and $M(u(X))$ is the (probabilistic) expected utility of X w.r.t. u .

In this section we will recall the definition of mixed expected utility and some of its properties. For the clarity of the presentation we will expose only the bidimensional case. Then a mixed vector will have the form (A, X) where A is a fuzzy number and X is a random variable. We will consider only the case (A, X) .

We fix a weighting function f and a bidimensional continuous utility function $u : \mathbf{R}^2 \rightarrow \mathbf{R}$. Let (A, X) be a mixed vector. Assume that the level sets of A have the form $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$, $\gamma \in [0, 1]$. For any $a \in \mathbf{R}$, $u(a, X) : \Omega \rightarrow \mathbf{R}$ will be the random variable defined by $u(a, X)(w) = u(a, X(w))$ for any $w \in \Omega$.

Definition 2.1 [13], [16] The mixed expected utility $E(f, u(A, X))$ associated with f, u and the mixed vector (A, X) is defined by

$$E(f, u(A, X)) = \frac{1}{2} \int_0^1 [M(u(a_1(\gamma), X)) + M(u(a_2(\gamma), X))] f(\gamma) d\gamma \quad (1)$$

Remark 2.2 (i) If the fuzzy number A is the constant a then $E(f, u(A, X)) = M(u(a, X))$.

(ii) If the random variable X is the constant b then $E(f, u(A, b)) = \frac{1}{2} \int_0^1 [u(a_1(\gamma), b) + u(a_2(\gamma), b)] f(\gamma) d\gamma$

The following two propositions are essential to prove the main theorems from the following section.

Proposition 2.3 [13], [16] Let g, h be two bidimensional utility functions and $a, b \in \mathbf{R}$. If $u = ag + bh$ then $E(f, u(A, X)) = aE(f, g(A, X)) + bE(f, h(A, X))$.

Proposition 2.4 [13], [16] If the utility function u has the form $u(x, y) = (x - E(f, A))(y - M(X))$ then $E(f, u(A, X)) = 0$.

3. Optimal saving with probabilistic background risk

The optimal saving models from [9], [10] regard the presence of two types of risk: income risk and background risk, both of them being mathematically represented by random variables. In this section we will present the general features of these models to serve as a starting point in the building of mixed models from the following sections.

The two-period models of [9], [10] are characterized by the following data:

- $u(y, x)$ and $v(y, x)$ are consumer's utility functions for period 0, resp. 1.
- the variable y represents the income, and x is a non-financial variable.
- for period 0, the variables y and x have the sure values y_0 and x_0 .
- for period 1, there is an uncertain income (described by the random variable Y) and a background risk (described by the random variable X).

We denote $\bar{y} = M(Y)$, $\bar{x} = M(X)$. In [10] the following possible situations on variables y and x are mentioned:

- $y = Y$, $x = X$ (income risk and background risk)
- $y = Y$, $x = \bar{x}$ (income risk and no background risk)
- $y = \bar{y}$, $x = X$ (background risk and no income risk)
- $y = \bar{y}$, $x = \bar{x}$ (no uncertainty)

Consider now the following expected lifetime utilities corresponding to situations (a), (c), (d):

$$V(s) = u(y_0 - s, x_0) + M(v(Y + s, X)) \quad (1)$$

$$W(s) = u(y_0 - s, x_0) + M(v(\bar{y} + s, X)) \quad (2)$$

$$T(s) = u(y_0 - s, x_0) + v(\bar{y} + s, \bar{x}) \quad (3)$$

where s is the level of saving. By [10], the optimization problem is formulated:

$$\max_s V(s) = V(s^*) \quad (4)$$

$$\max_s W(s) = W(s^\circ) \quad (5)$$

$$\max_s T(s) = T(s^{\circ\circ}) \quad (6)$$

with the optimal solutions $s^* = s^*(Y, X)$, $s^\circ = s^\circ(\bar{y}, X)$ and $s^{\circ\circ} = s^{\circ\circ}(\bar{y}, \bar{x})$.

The differences $s^* - s^\circ$, $s^* - s^{\circ\circ}$ are called *precautionary saving*, resp. *two-source precautionary saving*. In [10] necessary and sufficient conditions are given such that $s^* - s^\circ \geq 0$, resp. $s^* - s^{\circ\circ} \geq 0$, generalizing some results previously obtained in [9].

4. Mixed models of type I

The mixed models of this section are based on the hypothesis: the income risk is described by a fuzzy number A and the background risk is described by a random variable X . We will follow a parallel line with the one of [10] and we will keep the notations from the previous section. A fuzzy number A corresponds to the variable y and the random variable X corresponds to the variable x . Thus instead of the random vector (Y, X) we have a mixed vector (A, X) .

We fix a weighting function f and we denote $a = E(f, A)$ and $\bar{x} = M(X)$. In this case the situations (a)-(d) of Section 3 become

$$(a_1) \ y = A, x = X$$

$$(b_1) \ y = A, x = \bar{x}$$

$$(c_1) \ y = a, x = X$$

$$(d_1) \ y = a, x = \bar{x}$$

In this section we will study the way the optimal saving changes following the routes $(c_1) \rightarrow (a_1)$, $(d_1) \rightarrow (a_1)$ and $(b_1) \rightarrow (a_1)$. The first two are analogous to the cases studied in [10] for the probabilistic models. We will define three notions of "precautionary saving" and we will establish necessary and sufficient conditions for the positivity of these indicators.

Assume that the bidimensional utility functions u and v are strictly increasing w.r.t. each component, strictly concave and three times continuously differentiable. We denote u_i, u_{ij}, u_{ijk} (resp. v_i, v_{ij}, v_{ijk}) the first, the second and the third partial derivatives of u (resp. v).

Next we will use, as in [10], the following Taylor approximation:

$$v_1(y + s, x) \approx v_1(a + s, \bar{x})(x - \bar{x}) + v_{11}(a + s, \bar{x})(y - a) + v_{12}(a + s, \bar{x}) + \\ + \frac{1}{2}[v_{111}(a + s, \bar{x})(y - a)^2 + v_{122}(a + s, \bar{x})(x - \bar{x})^2 + 2v_{112}(a + s, \bar{x})(y - a)(x - \bar{x})] \quad (1)$$

Using the notion of mixed expected utility from Section 2, we introduce the following expected lifetime utilities:

$$V_1(s) = u(y_0 - s, x_0) + E(f, v(A + s, X)) \quad (2)$$

$$W_1(s) = u(y_0 - s, x_0) + E(f, v(a + s, X)) = u(y_0 - s, x_0) + M(v(a + s, X)) \quad (3)$$

$$T_1(s) = u(y_0 - s, x_0) + v(a + s, \bar{x}) \quad (4)$$

$$U_1(s) = u(y_0 - s, x_0) + E(f, v(A + s, \bar{x})) \quad (5)$$

V_1, W_1, T_1 are the analogues of V, W, T of Section 3, and U_1 comes from the situation (b_1) from above. By formula (1) from Section 1:

$$V_1(s) = u(y_0 - s, x) + (6) \\ + \frac{1}{2} \int_0^1 [M(v(a_1(\gamma) + s, X)) + M(v(a_2(\gamma) + s, X))] f(\gamma) d\gamma$$

By derivation, from (6) one obtains:

$$V_1'(s) = -u_1(y_0 - s, x_0) + \\ + \frac{1}{2} \int_0^1 [M(v_1(a_1(\gamma) + s, X)) + M(v_1(a_2(\gamma) + s, X))] f(\gamma) d\gamma$$

which, by formula (1) of Section 1 can be written

$$V_1'(s) = -u_1(y_0 - s, x_0) + E(f, v_1(A + s, X)) \quad (7)$$

By derivation, from (3)–(5) it follows:

$$W_1'(s) = -u_1(y_0 - s, x_0) + M(v_1(a + s, X)) \quad (8)$$

$$T_1'(s) = -u_1(y_0 - s, x_0) + v_1(a + s, \bar{x}) \quad (9)$$

$$U_1'(s) = -u_1(y_0 - s, x_0) + E(f, v_1(A + s, \bar{x})) \quad (10)$$

Proposition 4.1 *The functions V_1, W_1, T_1 and U_1 are strictly concave.*

We consider now the following optimization problems:

$$\max_s V_1(s) = V_1(s_1^*) \quad (11)$$

$$\max_s W_1(s) = W_1(s_1^\circ) \quad (12)$$

$$\max_s T_1(s) = T_1(s_1^{\circ\circ}) \quad (13)$$

$$\max_s U_1(s) = U_1(s_1^\Delta) \quad (14)$$

in which $s_1^* = s_1^*(A, X)$, $s_1^\circ = s_1^\circ(a, X)$, $s_1^{\circ\circ} = s_1^{\circ\circ}(a, \bar{x})$ and $s_1^\Delta = s_1^\Delta(A, \bar{x})$ are optimal solutions.

By Proposition 4.1, the four optimal solutions are given by:

$$V_1'(s_1^*) = 0, W_1'(s_1^\circ) = 0, T_1'(s_1^{\circ\circ}) = 0 \text{ and } U_1'(s_1^\Delta) = 0.$$

Taking into account (7)–(10), the optimal conditions are written:

$$u_1(y_0 - s_1^*, x_0) = E(f, v_1(f, v_1(A + s_1^*, X))) \quad (15)$$

$$u_1(y_0 - s_1^\circ, x_0) = M(v_1(a + s_1^\circ, X)) \quad (16)$$

$$u_1(y_0 - s_1^{\circ\circ}, x_0) = M(v_1(a + s_1^{\circ\circ}, X)) = v_1(a + s_1^{\circ\circ}, \bar{x}) \quad (17)$$

$$u_1(y_0 - s_1^\Delta, x_0) = E(f, v_1(A + s_1^\Delta, \bar{x})) \quad (18)$$

Following the line of [10], we will introduce three notions of "mixed precautionary saving": $s_1^* - s_1^\circ$, $s_1^* - s_1^{\circ\circ}$, $s_1^* - s_1^\Delta$.

$s_1^* - s_1^\circ$ corresponds to precautionary saving from [10] and measures the modification of the optimal saving by passing from $(y = a, x = X)$ to $(y = A, x = X)$, i.e. by adding the income risk A in the presence of the background risk X . The difference $s_1^* - s_1^{\circ\circ}$ expresses the modification of the optimal saving by passing from the certain situation $(y = a, x = \bar{x})$ to the situation $(y = A, x = X)$, i.e. by adding the income risk A and the background risk X . Finally, $s_1^* - s_1^\Delta$ measures the modification of the optimal saving by passing from $(y = A, x = \bar{x})$ to $(y = A, x = X)$, i.e. by adding the background risk X in the presence of the income risk A .

Next we intend to give necessary and sufficient conditions for the positivity of the three indicators.

Proposition 4.2 *Let (A, X) be a mixed vector with $a = E(f, A)$ and $\bar{x} = M(X)$. The following are equivalent:*

- (i) $s_1^*(A, X) - s_1^\circ(a, X) \geq 0$;
- (ii) $v_{111}(a + s_1^*(A, X), \bar{x}) \geq 0$.

The property (i) of the previous proposition says that the effect of adding the income risk A in the presence of background risk X is the increase of the optimal saving. In particular, from Proposition 4.2 it follows that if $v_{111} > 0$ then $s_1^*(A, X) - s_1^\circ(a, X) \geq 0$ for any income risk A and for any background risk X .

Next we study the change of the optimal saving on the route $(d_1) \rightarrow (a_1)$.

Proposition 4.3 *Let (A, X) be a mixed vector with $a = E(f, A)$ and $\bar{x} = M(X)$. The following are equivalent:*

- (i) $s_1^*(A, X) - s_1^{\circ\circ}(a, \bar{x}) \geq 0$
- (ii) $v_{111}(a + s_1^*(A, X), \bar{x})\text{Var}(f, A) + v_{122}(a + s_1^*(A, X), \bar{x})\text{Var}(X) \geq 0$.

Condition (i) of Proposition 4.3 says that the effect of adding the income risk A and the background risk X is the increase of the optimal saving. In particular, from Proposition 4.3 it follows that if $v_{111} \geq 0$ and $v_{122} \geq 0$ then for any mixed vector (A, X) we have $s_1^* - s_1^{\circ\circ} \geq 0$.

Corollary 4.4 *Assume that (A, X) is a mixed vector and $s_1^*(A, X) - s_1^\circ(a, X) \geq 0$. If $v_{122} \geq 0$ then $s_1^*(A, X) - s_1^{\circ\circ}(a, \bar{x}) \geq 0$, where $a = E(f, A)$ and $\bar{x} = M(X)$.*

Finally consider now the change of the optimal saving on the route $(b_1) \rightarrow (a_1)$.

Proposition 4.5 *Let (A, X) be a mixed vector with $a = E(f, A)$ and $\bar{x} = M(X)$. The following are equivalent:*

- (i) $s_1^*(A, X) - s_1^\Delta(A, \bar{x}) \geq 0$;
- (ii) $v_{122}(a + s_1^*(A, X), \bar{x}) \geq 0$.

Condition (i) of the previous proposition says that adding the background risk X in the presence of the income risk A leads to an increase of the optimal saving.

Corollary 4.6 *If $s_1^* - s_1^\circ \geq 0$ and $s_1^* - s_1^\Delta \geq 0$ then $s_1^* - s_1^{\circ\circ} \geq 0$.*

In the next example we will show that there exist mixed vectors (A, X) with $a = E(f, A)$ and $\bar{x} = M(X)$ and the utility functions u, v for which condition $s_1^*(A, X) - s_1^\circ(a, X) \geq 0$ does not imply $s_1^*(A, X) - s_1^{\circ\circ}(a, \bar{x}) \geq 0$.

Example 4.7 Let c, d be two real numbers such that $0 < c < d$. Let A be a fuzzy number with $a_1(\gamma) = c$, $a_2(\gamma) = d$ for any $\gamma \in [0, 1]$ and X the uniform repartition on $[c, d]$. It is known that $M(X) = \frac{c+d}{2}$ and $\text{Var}(X) = \frac{(c-d)^2}{12}$. A simple calculation shows that $E(f, A) = \frac{c+d}{2}$ and $\text{Var}(f, A) = \frac{(c-d)^2}{4}$. Then

$$v_{111}(y, x)\text{Var}(f, A) + v_{122}(y, x)\text{Var}(X) = \frac{(c-d)^2}{4} [v_{111}(y, x) + \frac{1}{3}v_{122}(y, x)] \quad (19)$$

Assume that the utility function v has the form:

$$v(y, x) = -\frac{1}{\alpha} e^{-\alpha y \frac{x^{1-\gamma}}{1-\gamma}} \text{ with } y \in \mathbf{R}, x > 0, \gamma > 0, \gamma \neq 1.$$

One notices that

$$v_{111}(y, x) = \alpha^2 e^{-\alpha y \frac{x^{1-\gamma}}{1-\gamma}}; v_{122}(y, x) = -\gamma e^{-\alpha y x^{-\gamma-1}}.$$

Then from (19) it follows:

$$v_{111}(y, x)\text{Var}(f, A) + v_{122}(y, x)\text{Var}(X) = \frac{(c-d)^2}{4} e^{-\alpha y [\frac{x^{1-\gamma}}{1-\gamma} \alpha^2 - \frac{\gamma}{3} x^{-\gamma-1}]} \quad (20)$$

One notices

$$\frac{x^{1-\gamma}}{1-\gamma} - \frac{\gamma}{3} x^{-\gamma-1} \geq 0 \text{ iff } \frac{x^{1-\gamma}}{1-\gamma} \alpha^2 \geq \frac{\gamma}{3} x^{-\gamma-1}$$

$$\text{iff } \frac{x^2}{1-\gamma} \alpha^2 \geq \frac{\gamma}{3}$$

$$\text{iff } \frac{1}{\gamma(1-\gamma)} \alpha^2 \geq \frac{1}{3x^2}$$

From (20) and these equivalences it follows:

$$v_{111}(y, x)\text{Var}(f, A) + v_{122}(y, x)\text{Var}(X) \geq 0 \text{ iff } \frac{\alpha^2}{\gamma(1-\gamma)} \geq \frac{1}{3x^2} \quad (21)$$

Replacing in (21) $y = a + s_1^*(A, X)$ and $x = \bar{x} = \frac{c+d}{2}$ and taking into account Proposition 4.3 one obtains:

$$s_1^*(A, X) - s_1^{\circ\circ}(a, \bar{x}) \geq 0 \text{ iff } \frac{\alpha^2}{\gamma(1-\gamma)} \geq \frac{4}{3(c+d)^2} \quad (22)$$

For $\gamma = \frac{3}{4}$ from (22) it follows

$$s_1^*(A, X) - s_1^{\circ\circ}(a, \bar{x}) \geq 0 \text{ iff } 4\alpha^2 \geq \frac{1}{(c+d)^2} \text{ iff } c + d \geq \frac{1}{2\alpha}.$$

Then, if $c + d < \frac{1}{2\alpha}$ we will have $s_1^*(A, X) - s_1^{\circ\circ}(a, \bar{x}) < 0$. On the other hand,

$$v_{111}(y, x) = \alpha^2 e^{-\alpha y \frac{x^{1-\gamma}}{1-\gamma}} = \alpha^2 e^{-\alpha y \frac{x^{\frac{1}{4}}}{\frac{1}{4}}} = 4\alpha^2 e^{-\alpha y x^{\frac{1}{4}}} < 0$$

thus by Proposition 4.2, $s_1^*(A, X) - s_1^{\circ}(a, X) \geq 0$.

In particular, the above example shows that the converse of Corollary 4.6 is not true.

5. Mixed models of type II

The mixed models of this section assume that the income risk is represented by a random variable Y and the background risk by a fuzzy number B . We keep the hypotheses of Section 4 on the bidimensional utility functions u and v .

We fix a weighting function f . We denote $M(y) = \bar{y}$ and $E(f, B) = b$. Similarly as in previous sections, we consider the following cases:

$$(a_2) y = Y, x = B$$

$$(b_2) y = Y, x = b$$

$$(c_2) y = \bar{y}, x = B$$

$$(d_2) y = \bar{y}, x = b$$

We analyze the way the optimal saving changes on the following three routes: $(c_2) \rightarrow (a_2)$, $(d_2) \rightarrow (a_2)$ and $(b_2) \rightarrow (a_2)$. For each of these three cases we will introduce a notion of "precautionary saving" and we will prove necessary and sufficient conditions for the positivity of these three indicators.

Corresponding to the cases $(a_2) - (d_2)$ we introduce four expected lifetime utilities:

$$V_2(s) = u(y_0 - s, x_0) + E(f, v(Y + s, B)) \quad (1)$$

$$W_2(s) = u(y_0 - s, x_0) + E(f, v(\bar{y} + s, B)) \quad (2)$$

$$T_2(s) = u(y_0 - s, x_0) + v(\bar{y} + s, b) \quad (3)$$

$$U_2(s) = u(y_0 - s, x_0) + E(f, v(Y + s, b)) \quad (4)$$

Deriving (1)–(4) it follows

$$V'_2(s) = -u_1(y_0 - s, x_0) + E(f, v_1(Y + s, B)) \quad (5)$$

$$W_2'(s) = -u_1(y_0 - s, x_0) + E(f, v_1(\bar{y} + s, B)) \quad (6)$$

$$T_2'(s) = -u_1(y_0 - s, x_0) + v_1(\bar{y} + s, b) \quad (7)$$

$$U_2'(s) = -u_1(y_0 - s, x_0) + E(f, v_1(Y + s, b)) \quad (8)$$

Similarly to the previous section, it is proved that V_2, W_2, T_2 and U_2 are strictly concave functions. We form the four maximization problems:

$$\max_s V_2(s) = V_2(s_2^*) \quad (9)$$

$$\max_s W_2(s) = W_2(s_2^\circ) \quad (10)$$

$$\max_s T_2(s) = T_2(s_2^{\circ\circ}) \quad (11)$$

$$\max_s U_2(s) = U_2(s_2^{\Delta}) \quad (12)$$

in which $s_2^* = s_2^*(Y, B)$, $s_2^\circ = s_2^\circ(\bar{y}, B)$, $s_2^{\circ\circ} = s_2^{\circ\circ}(\bar{y}, b)$ and $s_2^{\Delta} = s_2^{\Delta}(Y, b)$ are the optimal solutions.

By (5)–(8) the optimal conditions $V_2'(s_2^*) = 0$, $W_2'(s_2^\circ) = 0$, $T_2'(s_2^{\circ\circ}) = 0$ and $U_2'(s_2^{\Delta}) = 0$ will be written:

$$u_1(y_0 - s_2^*, x_0) = E(f, v_1(Y + s_2^*, B)) \quad (13)$$

$$u_1(y_0 - s_2^\circ, x_0) = E(f, v_1(\bar{y} + s_2^\circ, B)) \quad (14)$$

$$u_1(y_0 - s_2^{\circ\circ}, x_0) = v_1(\bar{y} + s_2^{\circ\circ}, b) \quad (15)$$

$$u_1(y_0 - s_2^{\Delta}, x_0) = E(f, v_1(Y + s_2^{\Delta}, b)) \quad (16)$$

We consider the following notions of precautionary saving: $s_2^* - s_2^\circ$, $s_2^* - s_2^{\circ\circ}$ and $s_2^* - s_2^{\Delta}$. The precautionary saving $s_2^* - s_2^\circ$ measures the change of the optimal saving on the route $(c_2) \rightarrow (a_2)$, $s_2^* - s_2^{\circ\circ}$ on the route $(d_2) \rightarrow (a_2)$ and $s_2^* - s_2^{\Delta}$ on the route $(b_2) \rightarrow (a_2)$.

The following three propositions offer necessary and sufficient conditions for the positivity of the three indicators.

Proposition 5.1 Let (Y, B) be a mixed vector with $\bar{y} = M(Y)$ and $b = E(f, B)$. The following are equivalent:

- (i) $s_2^*(Y, B) - s_2^\circ(\bar{y}, B) \geq 0$
- (ii) $v_{111}(\bar{y} + s_2^*(Y, B), b) \geq 0$.

Proposition 5.2 Let (Y, B) be a mixed vector with $\bar{y} = M(Y)$ and $b = E(f, B)$. The following are equivalent:

- (i) $s_2^*(Y, B) - s_2^{\circ\circ}(\bar{y}, b) \geq 0$
- (ii) $v_{111}(\bar{y} + s_2^*(Y, B), b) \text{Var}(Y) + v_{122}(\bar{y} + s_2^*(Y, B), b) \text{Var}(f, B) \geq 0$.

Proposition 5.3 Let (Y, B) be a mixed vector with $\bar{y} = M(Y)$ and $b = E(f, B)$. The following are equivalent:

- (i) $s_2^*(Y, B) - s_2^{\Delta}(Y, b) \geq 0$
- (ii) $v_{122}(\bar{y} + s_2^*(Y, B), b) \geq 0$.

Corollary 5.4 Let (Y, B) be a mixed vector with $\bar{y} = M(Y)$ and $b = E(f, B)$. If $s_2^*(Y, B) - s_2^\circ(\bar{y}, B) \geq 0$ and $s_2^*(Y, B) - s_2^{\Delta}(Y, b) \geq 0$ then $s_2^*(Y, B) - s_2^{\circ\circ}(\bar{y}, b) \geq 0$.

The proofs of Propositions 5.2 and 5.3 and Corollary 5.4 are similar to those of Propositions 4.3 and 4.5 and Corollary 4.6.

Similarly to Section 4 one proves that the converse of Corollary 5.4 is not true.

The positivity conditions of the three notions of precautionary saving $s_2^* - s_2^\circ$, $s_2^* - s_2^{\circ\circ}$ and $s_2^* - s_2^{\Delta}$ express the fact that on the routes $(c_2) \rightarrow (a_2)$, $(d_2) \rightarrow (a_2)$ and $(b_2) \rightarrow (a_2)$ the optimal saving increases. The above results characterize these conditions in terms of the third partial derivatives of v .

6. Conclusions

The optimal saving models of this paper combine methods of probability and possibility theory. For mixed two-period models the way the optimal saving changes is studied in two cases:

- (I) the income risk is a fuzzy number and the background risk is a random variable.
- (II) the income risk is a random variable and the background risk is a fuzzy number.

For each of the two types of models three notions of mixed precautionary saving have been introduced. These indicators measure the variation of the optimal saving as a result of adding income risk in the presence of background risk, adding background risk in the presence of income risk or simultaneously adding income risk and background risk

to a certain situation. The main results of the paper characterize the positivity of the three indicators, which indicates that by the mentioned modifications the level of optimal saving increases.

The mixed models of the paper follow a parallel line with the probabilistic model of [10], where the background risk and the income risk are random variables. It remains to study a purely possibilistic optimal saving model, in which both the income risk and the background risk are fuzzy numbers.

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